IB DIPLOMA PROGRAMME
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M07/5/MATHL/HP3/ENG/TZ1/XX

22077206

MATHEMATICS
HIGHER LEVEL
PAPER 3

Wednesday 16 May 2007 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions in one section only.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

Statistics and probability

1. *[Maximum mark: 18]*

A zoologist believes that the number of eggs laid in the Spring by female birds of a certain breed follows a Poisson law. She observes 100 birds during this period and she produces the following table.

Number of eggs laid	Frequency
0	10
1	19
2	34
3	23
4	10
5	4

(a) Calculate the mean number of eggs laid by these birds. *[2 marks]*

(b) The zoologist wishes to determine whether or not a Poisson law provides a suitable model.

(i) Write down appropriate hypotheses.

(ii) Carry out a test at the 1 % significance level, and state your conclusion. *[16 marks]*

2. *[Maximum mark: 12]*

The ten children in a class were each given two puzzles and the times taken, in seconds, to solve them were recorded as follows.

Child	A	B	C	D	E	F	G	H	I	J
Puzzle 1	66.3	71.9	62.8	69.8	64.6	74.9	68.8	72.6	70.4	74.2
Puzzle 2	64.8	71.6	59.9	68.1	66.0	72.4	67.7	70.9	69.8	74.6

It is claimed that, on average, a child takes the same time to solve each puzzle. Treating the data as matched pairs, use a two-tailed test at the 5 % significance level to determine whether or not this claim is justified. *[12 marks]*

3. *[Maximum mark: 9]*

The daily rainfall in a holiday resort follows a normal distribution with mean μ mm and standard deviation σ mm. The rainfall each day is independent of the rainfall on other days.

On a randomly chosen day, there is a probability of 0.05 that the rainfall is greater than 10.2 mm.

In a randomly chosen 7-day week, there is a probability of 0.025 that the **mean** daily rainfall is less than 6.1 mm.

Find the value of μ and of σ. *[9 marks]*

4. *[Maximum mark: 11]*

An urn contains 15 marbles, b of which are blue and $(15-b)$ are red. Peter knows that the value of b is either 5 or 9 but he does not know which. He therefore sets up the hypotheses

$$H_0: b=5, \ H_1: b=9.$$

To choose which hypothesis to accept, he selects 3 marbles at random without replacement. Let X denote the number of blue marbles selected. He decides to accept H_1 if $X \ge 2$ and to accept H_0 otherwise.

(a) State the name given to the region $X \ge 2$. *[1 mark]*

(b) Find the probability of making

(i) a Type I error;

(ii) a Type II error. *[10 marks]*

5. *[Maximum mark: 10]*

Let $X_1, X_2, \ldots, X_{20}$ be independent random variables each having a geometric distribution with probability of success p equal to 0.6.

Let $Y = \sum_{i=1}^{20} X_i$.

(a) Explain why the random variable Y has a negative binomial distribution. *[2 marks]*

(b) Find the mean and variance of Y. *[4 marks]*

(c) Calculate $P(Y = 30)$. *[4 marks]*

SECTION B

Sets, relations and groups

1. *[Maximum mark: 10]*

Let $a, b \in \mathbb{Z}^+$ and define $aRb \Leftrightarrow a^2 \equiv b^2$ (modulo 3).

(a) Show that R is an equivalence relation. *[6 marks]*

(b) Find all the equivalence classes. *[4 marks]*

2. *[Maximum mark: 14]*

Let $*$ be a binary operation defined on $\mathbb{R}$ as follows:

$$a * b = a + b - 1$$

(a) Determine whether or not the operation $*$ is commutative. *[2 marks]*

(b) Show that $\{\mathbb{R}, *\}$ is a group. *[12 marks]*

3. *[Maximum mark: 12]*

The permutations p_1 and p_2 of the integers $\{1, 2, 3, 4, 5\}$ are given by

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}; \; p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}.$$

(a) Find the order of p_1. *[4 marks]*

(b) (i) Find $p_2 p_1$, the composite permutation p_1 followed by p_2.

(ii) Determine whether or not p_1 and p_2 commute under composition of permutations. *[4 marks]*

(c) Find $(p_1^{\,2} p_2)^{-1}$. *[4 marks]*

4. *[Maximum mark: 18]*

The set S contains the four elements a, b, c, d. The groups $\{S, \circ\}$ and $\{S, \times\}$ have the following Cayley tables.

$\circ$	a	b	c	d
a	c	d	a	b
b	d	c	b	a
c	a	b	c	d
d	b	a	d	c

$\times$	a	b	c	d
a	c	a	d	b
b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

(a) For each group,

(i) state the identity,

(ii) find the order of each of the elements. *[6 marks]*

(b) Write down all the proper subgroups of

(i) $\{S, \circ\}$;

(ii) $\{S, \times\}$. *[4 marks]*

(c) Solve the equation $\left(a \circ (x \times x)\right) \times d = c$. *[8 marks]*

5. *[Maximum mark: 6]*

Let A and B be sets such that $A \cap B = A \cup B$. Prove that $A = B$. *[6 marks]*

SECTION C

Series and differential equations

1. *[Maximum mark: 10]*

(a) Use l'Hôpital's Rule to find

(i) $\lim_{x\to 1} \dfrac{\ln x^2}{x-1}$;

(ii) $\lim_{x\to 0} \dfrac{\tan^2 x}{1-\cos x}$. *[8 marks]*

(b) Giving a reason, state whether the following argument is correct or incorrect.

"Using l'Hôpital's Rule, $\lim_{x\to 3} \dfrac{x-3}{x^2-3} = \lim_{x\to 3} \dfrac{1}{2x} = \dfrac{1}{6}$." *[2 marks]*

2. *[Maximum mark: 8]*

Given that the Maclaurin series for $e^{\sin x}$ is $a+bx+cx^2+dx^3+\ldots$, find the values of a, b, c and d. *[8 marks]*

3. *[Maximum mark: 12]*

Consider the infinite series $\sum_{n=1}^{\infty} \dfrac{1}{n(n+2)}$.

(a) Show that the series is convergent. *[3 marks]*

(b) (i) Express $\dfrac{1}{n(n+2)}$ in partial fractions.

(ii) Hence find $\sum_{n=1}^{\infty} \dfrac{1}{n(n+2)}$. *[9 marks]*

4. *[Maximum mark: 16]*

(a) Use integration by parts to show that

$$\int \sin x \cos x \, \mathrm{e}^{-\sin x} \mathrm{d}x = -\mathrm{e}^{-\sin x}(1+\sin x) + C .$$ *[4 marks]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y\cos x = \sin x \cos x$.

(b) Find an integrating factor. *[3 marks]*

(c) Solve the differential equation, given that $y = -2$ when $x = 0$. Give your answer in the form $y = f(x)$. *[9 marks]*

5. *[Maximum mark: 14]*

Find the interval of convergence of the series $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right) x^n$. *[14 marks]*

SECTION D

Discrete mathematics

1. *[Maximum mark: 14]*

The weights of the edges in a simple graph *G* are given in the following table.

Vertices	A	B	C	D	E	F
A	-	4	6	16	15	17
B	4	-	5	17	9	16
C	6	5	-	15	8	14
D	16	17	15	-	15	7
E	15	9	8	15	-	18
F	17	16	14	7	18	-

(a) Use Prim's Algorithm, starting with vertex F, to find and draw the minimum spanning tree for *G*. Your solution should indicate the order in which the edges are introduced. *[12 marks]*

(b) Use your tree to find an upper bound for the travelling salesman problem for *G*. *[2 marks]*

2. *[Maximum mark: 16]*

(a) Use the Euclidean algorithm to find the greatest common divisor of 43 and 73. *[5 marks]*

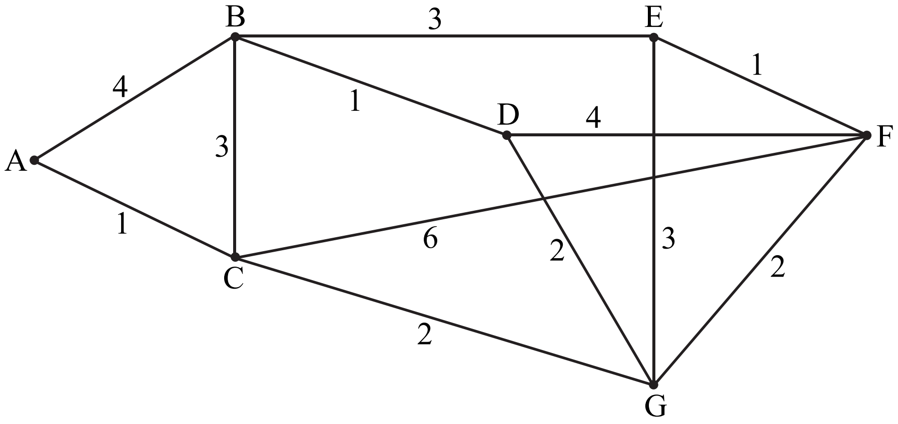
Consider the equation $43x + 73y = 7$, where $x,\ y \in \mathbb{Z}$.

(b) (i) Find the general solution of this equation.

(ii) Find the solution which minimises $|x| + |y|$. *[11 marks]*

3. *[Maximum mark: 13]*

Let H be the weighted graph drawn below.



(a) (i) Name the two vertices of odd degree.

(ii) State the shortest path between these two vertices.

(iii) Using the route inspection algorithm, or otherwise, find a walk, starting and ending at A, of minimum total weight which includes every edge at least once.

(iv) Calculate the weight of this walk. *[11 marks]*

(b) Write down a Hamiltonian cycle in H. *[2 marks]*

4. *[Maximum mark: 9]*

Consider the equation $x^{12}+1=7y$, where $x,\ y\in\mathbb{Z}^+$.
Using Fermat's little theorem, show that this equation has no solution. *[9 marks]*

5. *[Maximum mark: 8]*

Let K be a simple graph.

(a) Define the complement, K', of K. *[1 mark]*

(b) Given that K has six vertices, show that K and K' cannot both contain an Eulerian trail. *[7 marks]*